Matching of Asymptotic Expansions for a 2-D eigenvalue problem with two cavities linked by a narrow hole

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Abstract

One question of interest in an industrial conception of air planes motors is the study of the deviation of the acoustic resonance frequencies of a cavity which is linked to another one through a thin slot. These frequencies have a direct impact on the stability of the combustion in one of these two cavities. In this work, we aim is analyzing the eigenvalue problem for the Laplace operator with Dirichlet boundary conditions. Using the Matched Asymptotic Expansions technique, we derive the Asymptotic Expansion of this eigenmodes. Then, these results are validated through error estimates. Finally, we show how we can design a numerical method to compute the eigenvectors of this problem. The results are compared with direct computations.

1 The scientific context

In a turbo engine, the temperature of the combustion chamber can reach 2000 Kelvin. In order to protect the structure, small holes are perforated throw the wall linking the combustion chamber to the casing and fresh air is injected.

These small holes perturb the acoustic resonance frequencies and modes of the combustion chamber. This have often a negative impact on the combustion but a positive impact on the noise generated by the engine. The new environmental standard imposes a precise study of the effects of these small holes.

Unfortunately, a direct numerical approach is nowadays technically not feasible due to two main reasons.

- A fine mesh (in space and time due to the CFL condition) is compulsorily due to the small characteristic length of the holes.
- The mesh generation of a perforated structure is a hard job. This is mostly the case when the holes are numerous.

We aim in providing an efficient numerical method to take into account these small holes. The desired method should fulfils the following conditions

- mesh refinement is not required in the neighborhood of the slot.
- it must only involve quantities that can be easily computed.

Two natural approaches can be envisaged. The first one consists in replacing the effect of the wall by an equivalent transmission condition based on a surface homogenization technique, see for example [1]. The second approach consists in replacing each hole by an associated equivalent sources which intensity is derived by a multiscale analysis.

The experiments of physicist (see for example [2]) does not give a clear answer to which approach has to be considered. We have decided to approach this question with the equivalent sources point of view.

Moreover, the physical problem is really to complicated to be considered at this point. In the context of a 2-D toy model, we show that the so called technique of matching of asymptotic expansions (see for example [3] and [4]) permits to derive such an efficient method which can be interpreted as an equivalent point source model.

2 The toy model: A 2D eigenvalue problem

Let Ω_{int} and Ω_{ext} be two open subsets of $\mathbf{R}^2$ with

$$\begin{cases} \Omega_{int} \cap \Omega_{ext} = \emptyset \\ \exists a > 0 : \Big(\{0\} \times] - a; a[\Big) \in \partial\Omega_{int} \cap \partial\Omega_{ext}. \end{cases} \tag{1}$$

We consider the domain Ω^δ consisting of Ω_{ext} and Ω_{int} linked by a slit of width δ

$$\Omega^\delta := \Omega_{int} \cup \Omega_{ext} \cup \Big(\{0\} \times] - \frac{\delta}{2}; \frac{\delta}{2}[\Big) \subset \mathbf{R}^2 \tag{2}$$

which tends when $\delta \to 0$ to

$$\Omega := \Omega_{int} \cup \Omega_{ext} \subset \mathbf{R}^2. \tag{3}$$

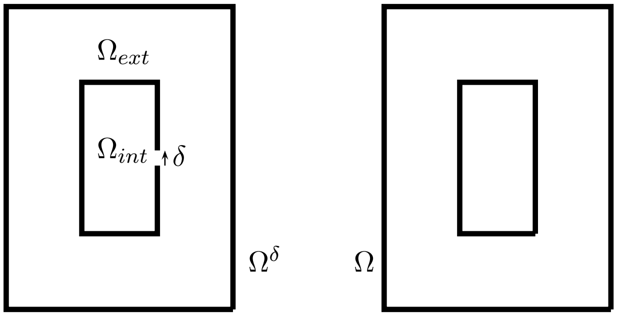

Figure 1: Geometry of the domain of propagation.

In these domains we consider the eigenvalue problems

$$\begin{cases} \text{Find } u^\delta \in \Omega^\delta \to \mathbf{R} \text{ and } \lambda^\delta \in \mathbf{R} \text{ satisfying} \\ -\Delta u^\delta(x,y) = \lambda^\delta u^\delta(x,y) \text{ in } \Omega^\delta, \\ u^\delta(x,y) = 0 \text{ on } \partial\Omega^\delta, \end{cases} \tag{4}$$

$$\begin{cases} \text{Find } u \in \Omega \to \mathbf{R} \text{ and } \lambda \in \mathbf{R} \text{ satisfying} \\ -\Delta u(x,y) = \lambda u(x,y) \text{ in } \Omega, \\ u(x,y) = 0 \text{ on } \partial\Omega, \end{cases} \tag{5}$$

that defines discreet sets of eigenmodes

- $(u_n^\delta, \lambda_n^\delta)_{n>0}$ which can be chosen to be a bi-orthogonal basis of $L^2(\Omega^\delta)$ and $H^1(\Omega^\delta)$ and to satisfy

$$\lambda_1^\delta \leq \lambda_2^\delta \leq ... \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n^\delta = +\infty. \tag{6}$$

- $(u_n, \lambda_n)_{n>0}$ which can be chosen to be a bi-orthogonal basis of $L^2(\Omega)$ and of $H^1(\Omega)$ and to satisfy

$$\lambda_1 \leq \lambda_2 \leq ... \quad \text{and} \quad \lim_{n\to+\infty} \lambda_n = +\infty. \tag{7}$$

Some natural question arises:

- Do the eigenvalue λ_n^δ converges to λ_n?
- Is it possible to obtain an asymptotic expansion of λ_n^δ?
- With this asymptotic expansion, is it possible to derive a numerical method to compute an approximation of λ_n^δ, with small computation cost?

3 The results

For the simplicity of the theoretical analysis, we assume that the eigenvalues $(\lambda_n)_{n>0}$, defined by (5), are simple ($\lambda_n = \lambda_p \implies p = n$).

Theorem *Let n be a strictly positive integer. The eigenvalue λ_n^δ can be expanded as follows if respectively $u_n = 0$ in Ω_{ext} or $u_n = 0$ in Ω_{int}*

$$\begin{aligned} \lambda_n^\delta &= \lambda_n - \frac{\pi}{16}\frac{|\partial_x u_n(0^-,0)|^2}{\|u_n\|^2_{L^2(\Omega)}}\delta^2 + \underset{\delta\to 0}{o}(\delta^2), \\ \lambda_n^\delta &= \lambda_n - \frac{\pi}{16}\frac{|\partial_x u_n(0^+,0)|^2}{\|u_n\|^2_{L^2(\Omega)}}\delta^2 + \underset{\delta\to 0}{o}(\delta^2). \end{aligned} \tag{8}$$

The condition of simplicity of the eigenvalues is to our opinion not central and is mostly considered for convenience to avoid resonance phenomena between two close eigenvalues of the Dirichlet-Laplacian in Ω^δ.

Moreover this condition implies that all the eigenvectors of the Dirichlet-Laplacian of Ω are eigenvectors of the Dirichlet-Laplacian of either Ω_{int} or of Ω_{ext}. Consequently they satisfy

$$u_n = 0 \text{ in } \Omega_{int} \text{ or in } \Omega_{int}. \tag{9}$$

When δ is small, the formula (8) provides a way to compute an approximation of the eigenvalue λ_n^δ involving only the computation of the eigenmodes of the Dirichlet-Laplacian in Ω. This implies that, for small $\delta > 0$, no mesh refinement is required to obtain a precise approximation of the eigenvalues of Ω^δ.

References

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